GLOBAL NONLINEAR STABILITY OF MINKOWSKI SPACETIME: AN INTRODUCTION

ZHENGJUN (JASPER) LIANG

Abstract. The purpose of this note is to give an exposition of the result [4] by Christoudoulous and Klainerman on global nonlinear stability of the Minkowski spacetime. We introduce the background, present an overview of the methods, and emphasize intuitions behind the ideas. Most of the materials are drawn from the introduction part of [4] and the relevant chapter on [3].

Contents

1. Preliminaries 1
2. Introduction 3
3. Acknowledgement 4
4. A Program for the Proof 4
5. The Setup: Asymptotically Flat Initial Data 5
5.1. Asymptotic Flatness 5
6. Conserved Quantities: Preservation of Symmetry 6
6.1. A Simple Example 6
6.2. Einstein Vacuum Equations 7
6.3. The Time Translations 8
6.4. The Space Rotations 8
6.5. The Conserved Quantities and Smallness Assumption on Initial Data 10
References 11

1. Preliminaries

Suppose we have a Lorentzian spacetime $(\mathbf{M}, \mathbf{g})$ with covariant differentiation $\mathbf{D}$.

Definition 1.1. Given vector fields X, Y, Z on $(\mathbf{M}, \mathbf{g})$, the **Riemannian curvature tensor** is defined by

$$\mathbf{R}(X,Y)Z := (\mathbf{D}_X\mathbf{D}_Y - \mathbf{D}_Y\mathbf{D}_X)Z - \mathbf{D}_{[X,Y]}Z$$

or, in coordinate form, we put

$$\mathbf{R}_{\gamma\sigma\alpha\beta} = \mathbf{g}\left(\mathbf{R}\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)\frac{\partial}{\partial x^\sigma}, \frac{\partial}{\partial x^\gamma}\right)$$

The **Ricci curvature tensor** is given by

$$\mathbf{R}_{\mu\nu} = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\mu\nu\alpha\beta}$$

and the **scalar curvature** is given by

$$\mathbf{R} = \mathbf{g}^{\mu\nu}\mathbf{R}_{\mu\nu}$$

Theorem 1.2 (Bianchi Identity)**.** *We have the* ***Bianchi identity***

$$\mathbf{D}_{[\varepsilon}\mathbf{R}_{\alpha\beta]\gamma\delta} := \frac{1}{3}(\mathbf{D}_\varepsilon\mathbf{R}_{\alpha\beta\gamma\delta} + \mathbf{D}_\alpha\mathbf{R}_{\beta\varepsilon\gamma\delta} + \mathbf{D}_\beta\mathbf{R}_{\varepsilon\alpha\gamma\delta}) = 0 \tag{1.3}$$

Definition 1.4. A hypersurface $H \subset \mathbf{M}$ is called spacelike if at each $x \in H$ the metric

$$\mathbf{g}_x|_{T_x H} =: g_x$$

is positive definite.

Definition 1.5 (Second Fundamental Form)**.** The second fundamental form k of $(H,\ g)$ is a 2-covariant, symmetric tensor field on H defined by

$$k(X, Y) = g(D_X N, Y)$$

for $X, Y \in T_x H$. Here N is the outward normal vector at $x \in H$.

Definition 1.6 (Cauchy Hypersurface)**.** A Cauchy hypersurface $H \subset \mathbf{M}$ is a complete, spacelike hypersurface such that any causal curve $\gamma \subset \mathbf{M}$ intersects H at most once.

A spacetime admitting a Cauchy hypersurface is called **globally hyperbolic**.

Definition 1.7 (Time Function)**.** A time function $t : \mathbf{M} \to [0, \infty]$ is a differentiable function such that

$$\langle dt, X \rangle > 0$$

for all $X \in I_p^+$ where $p \in \mathbf{M}$.

Topologically, a space-time foliated by the level surfaces of a time function is diffeomorphic to a product manifold $\mathbb{R} \times \Sigma$ where Σ is 3-dimensional. We also note that the space-time can be parametrized by by points on the slice $t = 0$ by following integral curves of $\mathbf{D}t$. Relative to this parametrization, the space-time metric takes the form

$$ds^2 = -\phi^2(t, x)dt^2 + \sum_{i,j=1}^{3} g_{ij}(t, x)dx^i dx^j \tag{1.8}$$

where $x = (x^1, x^2, x^3)$ are arbitrary coordinates on the slice $t = 0$.

Definition 1.9 (Lapse Function)**.** The function

$$\phi(t, x) = -\frac{1}{\langle \mathbf{D}t, \mathbf{D}t \rangle^{1/2}}$$

above is called the lapse function of the foliation.

The foliation is said to be normalized at infinity if $\phi \to 1$ as $x \to \infty$ on each leaf H_t.

Proposition 1.10. *Consider a frame (e_1, e_2, e_3) on H_t, we have*

$$k_{ij} := k(e_i, e_j) = \frac{1}{2\phi} \frac{\partial g_{ij}}{\partial t}$$

We also mention some basic facts about Lagrangians and Euler-Lagrange equations. Denote by x the the independent variables x^μ, $\mu = 1, ..., n$, by q the dependent variables q^a, $a = 1, ..., m$, and by v the first derivatives of dependent variables v_μ^α. Then the Lagrangian L is a given function

$$L = L(x, q, v)$$

Then a set of functions $(u^a(x) : a = 1, ..., m)$ is a solution of the Euler-Lagrange equations, if substituting

$$q^a = u^a(x)$$

$$v_\mu^\alpha = \frac{\partial u^a}{\partial x^\mu}(x)$$

We have

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v_\mu^a}(x, u(x), \partial u(x))\right) - \frac{\partial L}{\partial q^a}(x, u(x), \partial u(x)) = 0$$

There are advantages of formulating an equation as a Euler-Lagrange equation, since we have the following Noether's theorem that connects preservation of symmetry and conserved quantities.

Theorem 1.11 (Noether's Theorem). *In the framework of a Lagrangian theory, to each continuous group of transformations leaving the Lagrangian invariant there corresponds a quantity which is conserved.*

Remark 1.12. If we are on a Pseudo-Riemannian manifold $(\mathbf{M}, \mathbf{g})$, we sometimes say a Lagrangian L is of the form

$$L = L^* d\mu_{\mathbf{g}}$$

where $d\mu_{\mathbf{g}}$ is the volume form and L^* is a scalar function as above.

2. Introduction

We have the **Einstein Field Equations**

$$\boxed{\mathbf{G}_{\mu\nu} = 8\pi \mathbf{T}_{\mu\nu}} \tag{2.1}$$

where

$$\mathbf{G}_{\mu\nu} = \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R}$$

and $\mathbf{T}_{\mu\nu}$ is related to the presence of some matter fields. In the simplest situation of the physical vacuum, $\mathbf{T} = 0$. We then have

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = 0$$

Taking trace on both sides we have

$$\mathbf{R} = 0$$

so we get the **Einstein vacuum equations**

$$\boxed{\mathbf{R}_{\mu\nu} = 0} \tag{2.2}$$

which is our primary object of study.

Suppose that the space-time $(\mathbf{M}, \mathbf{g})$ can be foliated by the level sets of a time function t, we can write our space-time metric in the canonical form as in (1.8). The E-V equations are then equivalent to the following **constraint equations**

$$\nabla^j k_{ji} - \nabla_i \mathrm{tr} k = 0 \tag{2.3}$$

$$R - |k|^2 + (\mathrm{tr} k)^2 = 0 \tag{2.4}$$

and the **evolution equations**

$$\partial_t g_{ij} = -2\phi k_{ij} \tag{2.5}$$

$$\partial_t k_{ij} = -\nabla_i \nabla_j \phi + \phi(R_{ij} + \mathrm{tr} k k_{ij} - 2k_{ia}k_j^a) \tag{2.6}$$

where $i, j \in \{1, 2, 3\}$.

We first observe that the evolution equations have 13 unknowns while there are only 12 equations in total. This gives us the freedom to choose the time function.

Moreover, in view of Bianchi identities, if g and k satisfy the evolution equations, the constraint equations are automatically satisfied on any time slice H_t given that there is an initial slice H_{t_0} on which the given conditions for g and k are satisfied. We can thus formulate the E-V equations as a **Cauchy problem**. That is, given an initial data set (Σ, g, k) that satisfies the constraint equations, we want to find unknowns $(H_t, g(t), k(t))$ that solve the evolution equations and satisfy

$$(H_0, g(0), k(0)) = (\Sigma, g, k)$$

For convenience of notation, we usually suppress the dependence of g and k on time t.

The celebrated results of Choquet-Bruhat and Geroch give the local existence and maximum development of the Cauchy problem.

Theorem 2.7 (Choquet-Bruhat, Geroch, [1], [2]). *Any initial data set $(\overline{M}, \overline{g}, k)$ satisfying the constraint equations gives rise to a unique maximal development.*

The proof involves writing the (foliated) E-V equations under harmonic coordinates and showing that the problem can be reduced to showing local wellposedness of a nonlinear wave equation.

After a local existence result is proven, there are two natural questions one can ask:

(1) How long is the maximal time of existence? Can we prescribe a set of initial data that give rise to global existence?
(2) Are the solutions we obtain stable? That is, if two sets of initial data are very close to each other, will they stay close under time evolution?

The seminal work of Christoudoulou and Klainerman [4] gives an affirmative answer to the above questions. The goal of this note is to give an exposition of some elements involved in their proof.

We also want to mention that the work of Bieri [6] extends the above result to more general asymptotically flat initial data. Using harmonic coordinates, Lindblad and Rodnianski [5] also gave a proof of a similar stability result. The argument is less lengthy than in [4], while it also has a less precise description of the geometry.

3. Acknowledgement

This note serves as the final project to MATH 636, Topics in Differential Geometry: General Relativity, at the University of Michigan in Winter 2024. I appreciate professor Lydia Bieri for giving references and suggestions on this project. Her passionate and energetic delivery of lectures helped me learn a lot of these material.

4. A Program for the Proof

Before we state a rigorous version of the main theorem being proved, we give a heuristic version of the theorem and a program of the proof.

Theorem 4.1 (Global Nonlinear Stability of Minkowski Spacetime, Version 1). *Any "strongly asymptotically flat" initial data satisfying a suitable smallness assumption leads to a unique globally hyperbolic and geodesically complete development.*

Given a desirable set of initial data $\overline{I}$, let $t_{max} \in (0, \infty]$ be the time of maximal development to the Cauchy problem. At a very high level, the proof relies on a continuity or bootstrap argument roughly as follows.

Step 1: One designs a continuous in time quantity related to the E-V equations (2.2) and denote it by

$$Q(\overline{I}, t)$$

Here the notation $Q(\overline{I}, t)$ means that it depends on the initial data $\overline{I}$ and time $0 \leqslant t \leqslant t_{max}$. Given $\varepsilon_0 > 0$, we define

$$t_* = \max\{0 \leqslant t \leqslant t_{max} : Q(\overline{I}, t) \leqslant \varepsilon_0\}$$

and denote by

$$U_{t_*} := \bigcup_{0 \leqslant t \leqslant t_*} H_t$$

the spacetime slab up to t_*.

Step 2: Using energy estimates, one can show that if the initial data is sufficiently small in some suitable sense, one actually has

$$Q(\overline{I}, t) \leqslant \frac{\varepsilon_0}{2}$$

for $0 \leqslant t \leqslant t_*$. At the mean time, the data at t_* preserves essential structures satisfied by the initial data.

Step 3: Given the desirable traits of the data at time t_*, one can use it as a new set of "initial data", and extend U_{t_*} to $U_{t_*+\delta}$ for some $\delta > 0$. If $\delta > 0$ is small, we still have

$$Q(\overline{I}, t) \leqslant \varepsilon_0$$

up to $t_* + \delta$. If $t_* < \infty$, the maximality of t_* is contradicted. Thus $t_* = \infty$ and $t_{max} = \infty$, as desired.

5. The Setup: Asymptotically Flat Initial Data

5.1. **Asymptotic Flatness.** We first have to define the notion of strongly asymptotically flat initial data. Roughly speaking, an initial data set (Σ, g, k) is asymptotically flat if the complement of a finite (in volume) set in Σ is diffeomorphic to the complement of a ball in $\mathbb{R}^3$. On such initial data, we can define appropriate notions of energy, linear and angular momentum.

Definition 5.1 (S.A.F. Initial Data)**.** We say that an initial data set (Σ, g, k) satisfies the S.A.F. condition if g and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighborhood of infinity such that, as

$$r = \left[\sum_{i=1}^{3} (x^i)^2\right]^{1/2} \to \infty$$

we have

$$g_{ij} = (1 + 2M/r)\delta_{ij} + o_4(r^{-3/2}),$$
$$k_{ij} = o_3(r^{-5/2})$$

Remark 5.2. (1) A function f is said to be $o_m(r^{-k})$ resp. $O_m(r^{-k})$ as $r \to \infty$ if $\partial^l f = o(r^{-k-l})$ resp. $O(r^{-k-l})$ for any $l = 0, 1, ..., m$.
(2) We call the leading term in the expansion of g_{ij} the **Schwartzschild part** of the metric g.

Definition 5.3. We can define the ADM energy E, linear momentum P, and angular momentum J as follows:

$$E = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) N^j \, dA$$
$$P_i = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} (k_{ij} - (\mathrm{tr}k) g_{ij}) N^j \, dA, \quad i = 1, 2, 3$$
$$J_i = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \epsilon_{iab} x^a (k^{bj} - g^{bj} \mathrm{tr}k) N_j \, dA, \quad i = 1, 2, 3$$

Here N^j are components of the normal vector, and ϵ_{iab} are coefficients of the volume form with respect to an arbitrary frame.

We briefly explain where the definitions come from. First we have the following classical theorem of Noether (Theorem 1.11). The Einstein-Weyl Lagrangian is defined by

$$L = L^* d\mu_{\mathbf{g}}$$

where

$$L^* := -\frac{1}{4}\sqrt{-\mathbf{g}}\mathbf{g}^{\mu\nu}(\Gamma^\beta_{\mu\alpha}\Gamma^\beta_{\nu\alpha} - \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha})$$

and the Euler-Lagrange equation of the Einstein-Weyl Lagrangian gives rise to Einstein vacuum equations. Noether's theorem (1.11) tells us that we are able to get some conserved quantities out of transformations that leave the Einstein-Weyl Lagrangian invariant. On a space-time manifold that satisfies certain asymptotic flatness assumptions, one can thus define

(1) *Energy* that corresponds to time translations.
(2) *Linear momentum* that corresponds to space translations.
(3) *Angular momentum* that corresponds to space rotations.

Moreover, these quantities are geometric invariants, meaning that they do not depend on the underlying Riemmanian metric as long as it is asymptotically flat. Choosing appropriate coordinates, we can assume

$$\mathbf{g}_{0\nu} = 0$$

and we get the expressions in agreement with the ones in 5.3.

A more detailed discussion on these physical quantities can be found in [3].

6. Conserved Quantities: Preservation of Symmetry

One main difficulty in the proof is finding globally conserved quantities.

6.1. A Simple Example.

If we look at the linear wave equation in $\mathbb{R}^3$

$$\partial_t^2 u - \Delta u = 0 \tag{6.1}$$

we know that the energy

$$E(t) := \int_{\mathbb{R}^3} |\partial_t u|^2 \ dx + \int_{\mathbb{R}^3} |\nabla u|^2 \ dx \tag{6.2}$$

is conserved, since integration by parts yields

$$\frac{1}{2}\frac{dE(t)}{dt} = \int_{\mathbb{R}^3} \partial_t u \ (\partial_t^2 u - \Delta u) \ dx = 0$$

We also note that if u is a solution, then any derivative of u is also a solution, so we can define the energy $E(t)$ analogously for any derivative of u. Moreover, linear combinations of solutions is again a solution. This way we obtain many conserved quantities under this linear evolution.

When our equation is, for example, quasilinear, meaning that it has the form

$$\partial_t^2 u - a(u, Du)\Delta u = f(u, Du) \tag{6.3}$$

we can still define $E(t)$ as above for u and space derivatives of u, but we get

$$\frac{dE(t)}{dt} = \text{error terms} \tag{6.4}$$

In this situation, we want to exploit the symmetry of the equation and get more positive globally conserved quantities, so that we can control the error terms and bound

$$\frac{dE(t)}{dt} \leqslant C(t)E(t) + D(t)$$

where $C(t)$ and $D(t)$ only depend on the initial data and the globally conserved quantities.

6.2. **Einstein Vacuum Equations.** We begin by recalling a definition.

Definition 6.5 (Conformal Killing Vector Field)**.** A vector field X on $(\mathbf{M}, \mathbf{g})$ is conformal killing if

$$\mathcal{L}_X \mathbf{g} = \lambda \mathbf{g}$$

for some scalar function λ on $\mathbf{M}$. Conformal killing vector fields give rise to conformal isometries of the spacetime manifold.

In the context of Einstein vacuum equations, the analogy of derivatives is Lie derivatives with respect to conformal killing vector fields. That is, if Ψ solves the linearization of the E-V equations and X is a conformal killing field of the solution space-time, $\mathcal{L}_X \Psi$ is again a solution to the linearized E-V equations. Therefore, to close our argument, we want to find such conformal killing vector fields of the solution space-time to E-V equations to construct conserved positive integral quantities.

In implementing the idea above, one encounters a significant difficulty: a general space-time only has a trivial conformal isometry group, so there is no guarantee that one can find the desired conformal killing vector fields. However, spacetime arising from S.A.F. initial data is expected to approach the Minkowski spacetime in some sense as the time tends to infinity. The Minkowski spacetime does have a large conformal isometry group, so the hope is to define near time infinity the actions of the conformal group of Minkowski spacetime, and extend the actions backwards in time. The vector fields we get out of these actions are not exactly conformal killing, but we hope that their deformation tensors are globally small and tend suitably fast to 0 as $t \to \infty$.

We first recall the conformal group of the Minkowski spacetime.

(1) *Spacetime translations.* They are generated by the vector fields

$$T_\mu = \frac{\partial}{\partial x^\mu}, \quad \mu = 0, 1, 2, 3$$

(2) *Spacetime rotations.* They are generated by the vector fields

$$\Omega_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu}, \quad \mu, \nu = 0, 1, 2, 3, \ \mu < \nu$$

(3) *Scale transformations.* They are generated by the vector field

$$S = x^\mu \frac{\partial}{\partial x^\mu}$$

(4) *Inverted spacetime translations.* They are given by

$$I(\tilde{x}) = \frac{\tilde{x}}{(\tilde{x}, \tilde{x})}$$

where

$$(x, x) = \eta_{\alpha\beta} x^\alpha x^\beta$$

They are generated by the vector fields

$$K_\mu = 2x^\mu S + (x, x) T_\mu$$

When it comes to our strategy, it turns out that one has to restrict attention to a subgroup of the conformal group above due to some technicalities. Those are

(1) The time translations.
(2) The scale transformations.
(3) The inverted time translations.
(4) The spatial rotation group $O(3)$.

We briefly talk about how to define actions corresponding to (1) and (4) in our solution space-time to the E-V equations, which are the major difficulties.

6.3. **The Time Translations.** Remember that the 1 degree of freedom in the evolution equations allows us to choose the time function, and the group of time translations corresponds to the choice of a canonical maximum time function t. In fact, an inappropriate choice of time function might lead to a finite-time breakdown. To see this, suppose we choose the time function t such that the Lapse function

$$\Phi \equiv 1$$

Taking trace of (2.6) and using (2.4), we have

$$\partial_t \mathrm{tr}k = -\Delta\phi + \phi(R + (\mathrm{tr}k)^2) = |k|^2 \geqslant (\mathrm{tr}k)^2 \tag{6.6}$$

showing that tr$k$ will blow up in finite time.

Definition 6.7 (Maximum Time Function)**.** A maximal time function is a time function t whose level sets are maximal spacelike hypersurfaces which are complete and tend to parallel spacelike coordinate hyperplanes at spatial infinity. Here maximal means that the volume of the hypersurface is maximized among all compact perturbations of it. Moreover, we require the associated lapse function Φ to tend to 1 at spatial infinity.

The maximality condition is satisfied by imposing

$$\mathrm{tr}k = 0 \tag{6.8}$$

We in addition impose the canonical assumption

$$P_i = 0 \tag{6.9}$$

at all time, so that t is fixed up to addition of a constant. In fact, we only need to impose (6.8) and (6.9) on the initial data, since it will be preserved under the evolution equations.

Let H_t be the level surface at time t, we define the time translation subgroup $\{f_\tau\}_\tau$ where f_τ is a diffeomorphism between H_t and $H_{t+\tau}$. As we have seen above,

$$\boxed{T := \frac{\partial}{\partial x^0}} \tag{6.10}$$

is a generating vector field of such action.

6.4. **The Space Rotations.** The rotation subgroup $O(3)$ is defined to satisfy the condition that it takes any hypersurface H_t to itself. We use the following figure to illustrate what we do before we proceed to a more rigorous discussion.

We begin our construction by introducing the optical function u, which is a solution of the *Eikonal equation*

$$g^{\mu\nu}\partial_\mu u \partial_\nu u = 0 \tag{6.11}$$

The significance of the Eikonal equation is that the level surfaces C_u of u are null hypersurfaces. For convenience, we denote

$$a = -\frac{1}{\langle Du, Du\rangle^{1/2}}$$

The 2-surfaces of intersection

$$S_{t,u} := H_t \cap C_u$$

shall be orbits of the rotation subgroup $O(3)$ on each H_t.

Let $(H_{t_*}, \overline{g})$ be the final maximal hypersurface. We consider the vector field U on H_{t_*} given in local coordinates by

$$U^i = a^2 \overline{g}^{ij} \partial_j u$$

and note that the integral curves of U are orthogonal to $S_{t_*,u}$. Let $\{\varphi_\sigma\}_\sigma$ be the 1-parameter subgroup generated by U, we have

Proposition 6.12. *φ_σ restricts to a diffeomorphism of $S_{t_*,u}$ onto $S_{t_*,u+\sigma}$. In particular,*

$$\varphi_u : S_{t_*,0} \to S_{t_*,u}$$

is a diffeomorphism.

Next, we introduce a metric on $S_{t_*,0}$ that is pulled back from "the sphere at infinity".

Proposition 6.13. *Let γ be the induced metric on $S_{t_*,u}$ by the solution spacetime. The pullback to $S_{t_*,0}$ of γ rescaled by r^{-2}, namely*

$$\varphi_u^*(r^{-2}\gamma)_{S_{t_*,u}}$$

converges, as $u \to -\infty$ (the spacelike infinity on H_{t_*}), to a metric $\mathring{\gamma}_{t_*}$ with Gauss curvature equal to 1. Hence $(S_{t_*,0}, \mathring{\gamma}_{t_*})$ is isometric to S^2.

Therefore, the definition of the $O(3)$ action can be defined on $(S_{t_*,0}, \mathring{\gamma}_{t_*})$. We can then proceed to define the action on H_{t_*} by conjugation. Given $p \in S_{t_*,u}$ and $O \in O(3)$, we consider the integral curve

$$\sigma \mapsto \varphi_\sigma(p)$$

of U through p. As $\sigma \to -\infty$, it tends to some q which can be identified as a point on $(S_{t_*,0}, \mathring{\gamma}_{t_*})$. Then we can define the action

$$Op := \varphi_u(Oq) \in S_{t_*,u}$$

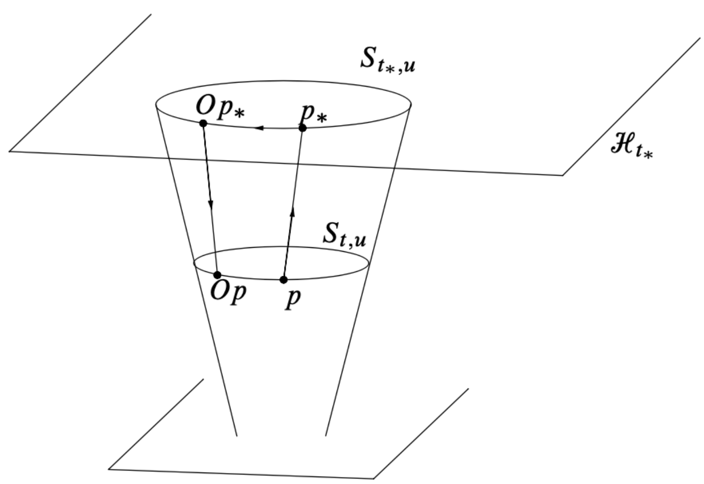

FIGURE 1. Extending the action to H_t, figure drawn from [3]

The next step is extending this action to the spacetime slab

$$U_{t_*} := \bigcup_{0 \leqslant t \leqslant t_*} H_t$$

For $p \in S_{t,u}$, let $L \in T_p C_u$ be the outgoing normal vector at p normalized by requiring that its component on T is equal to T. We define the group action on H_t for $0 \leqslant t \leqslant t_*$ by conjugation with the flow of L.

Lastly, we define the vector fields

$$\{\Omega^{(a)} : a = 1, 2, 3\}$$

that generate group actions by $O(3)$ and satisfy

$$[L, \Omega^{(a)}] = 0$$

and

$$[\Omega^{(a)}, \Omega^{(b)}] = \epsilon_{abc}\, \Omega^{(c)}$$

For this, let $p \in S_{t*,u}$. We pick a basis

$$\boxed{\{\Omega_0^{(a)} : a = 1, 2, 3\}} \tag{6.14}$$

of the Lie algebra of $O(3)$ such that

$$[\Omega_0^{(a)}, \Omega_0^{(b)}] = \epsilon_{abc}\, \Omega_0^{(c)}$$

We then identify them as vectors in $S_{t*,u}$ and flow them first along φ_u and then along L. These vector fields will play an important role in defining the aforementioned conserved quantities.

Remark 6.15. The optical function u is also used to define the vector fields S and V generating the scale transformations and inverted time translations, respectively.

6.5. **The Conserved Quantities and Smallness Assumption on Initial Data.** The quantity $Q(\bar{I}, t)$ we mentioned above will involve in particular the quantities

$$Q_1(t) := \int_{H_t} Q(\hat{\mathcal{L}}_O \mathbf{R})(\bar{K}, \bar{K}, T, T) + \int_{H_t} Q(\hat{\mathcal{L}}_T \mathbf{R})(\bar{K}, \bar{K}, \bar{K}, T) \tag{6.16}$$

$$\begin{aligned} Q_2(t) := &\int_{H_t} Q(\hat{\mathcal{L}}_O^2 \mathbf{R})(\bar{K}, \bar{K}, T, T) + \int_{H_t} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \mathbf{R})(\bar{K}, \bar{K}, \bar{K}, T) \\ &+ \int_{H_t} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \mathbf{R})(\bar{K}, \bar{K}, \bar{K}, T) + \int_{H_t} Q(\hat{\mathcal{L}}_T^2 \mathbf{R})(\bar{K}, \bar{K}, \bar{K}, T) \end{aligned} \tag{6.17}$$

which are conserved under the evolution of linearized Einstein vacuum equation.

Here T is a generator of time translations, S is a generator of scaling, K is a generator of inverted time translations,

$$\bar{K} = K + T$$

and also

$$Q(\hat{\mathcal{L}}_O \mathbf{R}) = \sum_1^3 Q(\hat{\mathcal{L}}_{\Omega^{(a)}} \mathbf{R})$$

Q is the Bel-Robinson tensor associated with $\mathbf{R}$. It is commonly used in general relativity to define local energy, and it satisfies the following properties:

1. It is fully symmetric and traceless.
2. It satisfies the positive energy condition, namely $Q(X, Y, Z, I)$ is positive whenever X, Y, Z, I are future-directed timeline vectors.
3. It is divergence-free, namely

$$\mathbf{D}^\delta Q_{\alpha\beta\gamma\delta} = 0$$

Eventually, we state the quantity related to the smallness assumption on the initial data mentioned above. On our initial hypersurface H_0, we define

$$\begin{aligned} Q(x_0, b) = &\sup_{H_0} \{b^{-2}(d_0^2 + b^2)^3 |\mathrm{Ric}|^2\} \\ &+ b^{-3}\left\{ \int_{H_0} \sum_{l=0}^3 (d_0^2 + b^2)^{l+1} |\nabla^l k|^2 + \int_{H_0} \sum_{l=0}^1 (d_0^2 + b^2)^{l+3} |\nabla^l B|^2 \right\} \end{aligned}$$

Here

$$d_0(x) = d(x_0, x)$$

is the Riemannian geodesic distance between the point x and x_0, b is a positive constant,

$$|\mathrm{Ric}|^2 = R^{ij} R_{ij}$$

and B, called the Bach tensor, is the symmetric, traceless 2-tensor given by

$$B_{ij} = \epsilon_j^{ab}\nabla_a(R_{ib} - 1/4 g_{ib}R)$$

where ϵ_j^{ab} are coefficients of the volume form.

Definition 6.18 (Global Smallness Assumption)**.** A complete metric g satisfies the global smallness assumption if there exists a sufficiently small $\varepsilon_0 > 0$ such that

$$\inf_{x_0 \in H_0, b \geqslant 0} Q(x_0, b) < \varepsilon_0$$

We give a rough idea of how the global smallness assumption gets used in the bootstrap argument. In particular, we want to estimate the error integral of deformation tensors of the approximate killing fields we constructed, and convince ourselves that the errors are sufficiently small. Let $\varepsilon_0 > 0$ and $Q(\bar{I}, t)$ be as above. One shows that

$$|\text{Error integrals of deformation tensors}| \leqslant C(\varepsilon_0)Q(\bar{I}, t)$$

where $C(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$, and obtains

$$Q(\bar{I}, t) \leqslant C' D(\bar{I}) + C(\varepsilon_0)Q(\bar{I}, t)$$

Taking $\varepsilon_0 > 0$ small enough, one has

$$Q(\bar{I}, t) \leqslant C' D(\bar{I})$$

which controls $Q(\bar{I}, t)$ in terms of initial data.

Eventually, we have the following more precise version of the global nonlinear stability theorem.

Theorem 6.19 (Global Nonlinear Stability of Minkowski Spacetime, Version 2)**.** *Any strongly asymptotically flat, maximal initial data set that satisfies the global smallness assumption stated above leads to a unique, globally hyperbolic, smooth, and geodesically complete solution of the E-V equations, which is foliated by a normal maximal time function t.*

References

[1] Y. Choquet-Bruhat, Théorème d'existence pour certain systèmes d'equations aux dérivées partielles nonlinéaires. *Acta Math.* **88** (1952), 141–225.

[2] Y. Choquet-Bruhat, R. P. Geroch, Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.* **14** (1969), 329–335.

[3] D. Christoudoulous, *Mathematical Problems of General Relativity I.* European Mathematical Society, 2008.

[4] D. Christoudoulous, S. Klainerman, *The Global Nonlinear Stability of Minkowski Space.* Princeton University Press, 1993.

[5] H. Lindblad, I Rodnianski. The Global Stability of Minkowski Space-Time in Harmonic Gauge. *Annals of Mathematics*, **171** (2010), 1401–77.

[6] L. Bieri, N. Zipser *Extensions of the Stability Theorem of the Minkowski Space in General Relativity.* The American Mathematical Society and International Press. 2009.